

The Fourier transform and convolutions generated by a differential operator with boundary condition on a segment

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Abstract. We introduce the concepts of the Fourier transform and convolution generated by an arbitrary restriction of the differentiation operator in the space $L_2(0, b)$. In contrast to the classical convolution, the introduced convolution explicitly depends on the boundary condition that defines the domain of the operator L . The convolution is closely connected to the inverse operator or to the resolvent. So, we first find a representation for the resolvent, and then introduce the required convolution.

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0. Introduction

The standard Fourier transform is a unitary transform in the Hilbert space $L_2(-\infty, +\infty)$ and it is generated by the operator of differentiation $(-i\frac{d}{dx})$, because the system of exponents $\{\exp(i\lambda x), \lambda \in R\}$ is a system of "eigenfunctions" corresponding to its continuous spectrum. The Fourier transform is closely connected to the bilinear, commutative, associative convolution without annihilators. An important fact is that the convolution with the fundamental solution allows us to find solutions of the inhomogeneous differential equation, which commutes with differentiation. Corresponding constructions can be generalized to arbitrary self-adjoint operators. Instead of the differential operator $(-i\frac{d}{dx})$ in the space $L_2(-\infty, +\infty)$, consider an operator L in the Hilbert space $L_2(0, b)$, where $b < \infty$, which is generated by the differential operator $(-i\frac{d}{dx})$ and a boundary condition. We introduce the concepts of the Fourier transform and convolution generated by an arbitrary restriction of the differentiation operator in the space $L_2(0, b)$. In contrast to the classical convolution, the introduced convolution explicitly depends on the boundary condition that defines the domain of the operator L .

As noted above, the convolution is closely connected to the inverse operator or to the resolvent. So, we first find a representation for the resolvent, and then introduce the required convolution.

1. Resolvent and spectrum of the operator L

Without loss of generality, we assume that the origin belongs to the resolvent set of the operator L , that is, there is an inverse operator L^{-1} . By M. Otelbaev's theorem [3] such operators are parameterized by a "boundary" function $\sigma(x)$ from the space $L_2(0, b)$.

Theorem 1.1 Let the action of the linear operator L in $L_2(0, b)$ be defined by formula $Ly = -i\frac{dy}{dx}$ with some (fixed) boundary condition. Suppose there exists the inverse operator L^{-1} in $L_2(0, b)$. Then there is a unique function $\sigma(x) \in L_2(0, b)$ such that the domain of operator L is

$$D(L) = \{y \in W_2^1[0, b] : y(0) - \int_0^b (-i\frac{dy}{dx})\overline{\sigma(x)}dx = 0\}.$$

Proof. Let us consider equation $Ly = f$, where $f \in L_2(0, b)$. Since there is the bounded inverse operator L^{-1} . We have $y = L^{-1}f$. Each solution of the differential equation $-i\frac{dy}{dx} = f$ has the form $y = c + K^{-1}f$, where c is an arbitrary constant and the operator K corresponds to the Cauchy problem with zero condition at zero:

$$Ky = -i\frac{dy}{dx}, D(K) = \{y \in W_2^1[0, b] : y(0) = 0\}.$$

Therefore, the constant $c = L^{-1}f - K^{-1}f$ is dependent on f and represents the value of a bounded linear functional on the Hilbert space $L_2(0, b)$. Then $c = c(f)$, and by the Riesz theorem on bounded linear functionals on $L_2(0, b)$, we have

$$c = \int_0^b f(x)\overline{\sigma(x)}dx, \sigma(x) \in L_2(0, b).$$

The element $\sigma(x)$ is uniquely determined. So, the solutions of the operator equation $Ly = f$ have the form $y = \int_0^b f(x)\overline{\sigma(x)}dx + (K^{-1}f)(x)$. In the last equation, we substitute $x = 0$. As a result, we get

$$y(0) = \int_0^b f(x)\overline{\sigma(x)}dx.$$

The converse assertion is also true and can be verified directly.

If the function $y(x)$ from $W_2^1[0, b]$ satisfies condition

$$U(y) = 0,$$

then it will belong to the domain D_L of L , where

$$U(y) := y(0) - \int_0^b (-i\frac{dy}{dx})\overline{\sigma(x)}dx.$$

Let us denote by $\Delta(\lambda)$ the entire function $\Delta(\lambda) = 1 - \lambda \int_0^b \exp(i\lambda x)\overline{\sigma(x)}dx$. Then the resolvent of the operator L is

$$(L - \lambda I)^{-1}f = i \int_0^x \exp(i\lambda(x - \xi))f(\xi)d\xi +$$

$$+\frac{\exp(i\lambda x)}{\Delta(\lambda)}\left(\int_0^b f(x)\overline{\sigma(x)}dx + \lambda i \int_0^b \overline{\sigma(x)}dx \int_0^x \exp(i\lambda(x-\xi))f(\xi)d\xi\right). \quad (1)$$

Indeed, denoting the right-hand side of this equality by $y(x)$, we find it is a direct consequence of

$$y'(x) = if(x) + i\lambda y(x).$$

Let us calculate

$$\begin{aligned} U(y) &= y(0) - \int_0^b (-i\frac{dy}{dx})\overline{\sigma(x)}dx = y(0) - \lambda \int_0^b y(x)\overline{\sigma(x)}dx - \int_0^b f(x)\overline{\sigma(x)}dx = \\ &= \frac{1}{\Delta(\lambda)}\left(\int_0^b f(x)\overline{\sigma(x)}dx + \lambda i \int_0^b \overline{\sigma(x)}dx \int_0^x \exp(i\lambda(x-\xi))f(\xi)d\xi - \right. \\ &\quad \left. - \lambda \int_0^b \overline{\sigma(x)}dx [i \int_0^x \exp(i\lambda(x-\xi))f(\xi)d\xi + \right. \\ &\quad \left. + \frac{\exp(i\lambda x)}{\Delta(\lambda)}\left(\int_0^b f(x)\overline{\sigma(x)}dx + \lambda i \int_0^b \overline{\sigma(x)}dx \int_0^x \exp(i\lambda(x-\xi))f(\xi)d\xi\right] - \int_0^b f(x)\overline{\sigma(x)}dx = \right. \\ &= \frac{1}{\Delta(\lambda)} \int_0^b f(x)\overline{\sigma(x)}dx + \frac{i\lambda}{\Delta(\lambda)} \int_0^b \overline{\sigma(x)}dx \int_0^x \exp(i\lambda(x-\xi))f(\xi)d\xi - \\ &\quad - \frac{i\lambda}{\Delta(\lambda)} \int_0^b \overline{\sigma(x)}dx \int_0^x \exp(i\lambda(x-\xi))f(\xi)d\xi + \\ &\quad + \lambda^2 i \int_0^b \overline{\sigma(x)}dx \int_0^x \exp(i\lambda(x-\xi))f(\xi)d\xi \int_0^b \exp(i\lambda\mu)\overline{\sigma(\mu)}d\mu - \\ &\quad - \lambda \frac{\exp(i\lambda x)}{\Delta(\lambda)} \int_0^b \overline{\sigma(x)}dx \int_0^x f(\xi)\overline{\sigma(\xi)}d\xi - \\ &\quad - \lambda^2 i \frac{\exp(i\lambda x)}{\Delta(\lambda)} \int_0^b \overline{\sigma(x)}dx \int_0^b \overline{\sigma(\mu)}d\mu \int_0^\mu \exp(i\lambda(\mu-\xi))f(\xi)d\xi - \frac{1}{\Delta(\lambda)} \int_0^b f(x)\overline{\sigma(x)}dx + \\ &\quad + \frac{\lambda}{\Delta(\lambda)} \int_0^b f(\xi)\overline{\sigma(\xi)}d\xi \int_0^b \exp(i\lambda x)\overline{\sigma(x)}dx = 0. \end{aligned}$$

The proof is complete.

From the representation of the resolvent by definition of the spectrum of operator L we get the following theorem.

Theorem 1.2 The set of zeros with multiplicities of the entire function $\Delta(\lambda)$ is exactly the same as the spectrum of the operator L .

Since $\Delta(\lambda)$ is the entire function in λ , we have that the spectrum of the operator L consists of isolated eigenvalues of finite multiplicity, and limit points of the spectrum can only be infinity. From the Paley – Wiener theorem it immediately follows that:

Theorem 1.3 The operator L has either countable number of eigenvalues, or they are absent. The spectrum is empty if and only if there exists constant $c \in [0, b]$ such that $\sigma(x) = i$ for $0 \leq x \leq c$ and $\sigma(x) = 0$ for $c \leq x \leq b$.

The proof immediately follows from the representation

$$\Delta(\lambda) = \exp(i\lambda c) - \int_0^c \exp(i\lambda x) \overline{(\sigma(x) - i)} dx - \lambda \int_c^b \exp(i\lambda x) \overline{\sigma(x)} dx$$

and from the Paley–Wiener theorem, since the presence of the integral term in the right side of this equation leads to the existence of the growing product $\exp(i\lambda c) \cdot \Delta(\lambda)$.

In what follows, suppose that the indicator diagram of the entire function $\Delta(\lambda)$ is the segment $[0, ib]$. Then the spectrum of the operator L is a countable set.

In order for the indicator diagram $\Delta(\lambda)$ to be represented by the interval $[0, ib]$, it is necessary and sufficient that

$$\min(\text{supp}(\sigma(x) - i)) = 0, \max(\text{supp}(\sigma(x))) = b, \quad (2)$$

where $\text{supp}(g)$ is the support of g .

The following theorem is proved just as in the work E. Titmarsh [4].

Theorem 1.4 Let the condition (2) hold. Then the number of zeros $N(r)$ of the function $\Delta(\lambda)$, which satisfy the inequality $|\lambda| \leq r$, satisfies the limit inequality

$$\lim_{r \rightarrow \infty} \frac{N(r)}{r} = \frac{b}{\pi}.$$

In the work of M. Cartwright [5] it is shown that if $\sigma(x)$ satisfies (2) and it is a function of a bounded variation, then all the zeros of the function $\Delta(\lambda)$ are in a horizontal strip and the value of $N_{\frac{b}{\pi}}(r)$ is bounded uniformly by r .

2. Convolution, generated by operator L

To obtain a convolution, we rewrite the resolvent $(L - \lambda I)^{-1}$ in the form

$$\begin{aligned} (L - \lambda I)^{-1} f &= i \int_0^x \frac{\exp(i\lambda(x - \xi))}{\Delta(\lambda)} f(\xi) d\xi \\ &- \int_0^b \overline{\sigma(\mu)} d\mu \frac{\partial}{\partial \mu} \left(\int_{\mu}^x \frac{\exp(i\lambda(x - \xi + \mu))}{\Delta(\lambda)} f(\xi) d\xi \right). \end{aligned} \quad (3)$$

Indeed, from the previous view of the resolvent (1), we have the chain of equalities:

$$\begin{aligned} (L - \lambda I)^{-1} f &= i \int_0^b \exp(i\lambda(x - \xi)) f(\xi) d\xi + \\ &+ \frac{\exp(i\lambda x)}{\Delta(\lambda)} \left(\int_0^b f(x) \overline{\sigma(x)} dx + \lambda i \int_0^b \overline{\sigma(x)} dx \int_0^x \exp(i\lambda(x - \xi)) f(\xi) d\xi \right) = \\ &= i \int_0^x \frac{\exp(i\lambda(x - \xi))}{\Delta(\lambda)} f(\xi) d\xi - \int_0^b \overline{\sigma(\mu)} d\mu \int_0^x \frac{\exp(i\lambda(x - \xi + \mu))}{\Delta(\lambda)} f(\xi) d\xi + \\ &+ \int_0^b \frac{\exp(i\lambda x)}{\Delta(\lambda)} f(\xi) \overline{\sigma(\xi)} d\xi + \lambda i \int_0^b \overline{\sigma(\mu)} d\mu \int_0^{\mu} \frac{\exp(i\lambda(x - \xi + \mu))}{\Delta(\lambda)} f(\xi) d\xi = \end{aligned}$$

$$\begin{aligned}
&= i \int_0^x \frac{\exp(i\lambda(x-\xi))}{\Delta(\lambda)} f(\xi) d\xi - \int_0^b \overline{\sigma(\mu)} d\mu \int_\mu^x \frac{\exp(i\lambda(x-\xi+\mu))}{\Delta(\lambda)} f(\xi) d\xi + \\
&\quad + \int_0^b \frac{\exp(i\lambda x)}{\Delta(\lambda)} f(\xi) \overline{\sigma(\xi)} d\xi = \\
&= i \int_0^x \frac{\exp(i\lambda(x-\xi))}{\Delta(\lambda)} f(\xi) d\xi - \int_0^b \overline{\sigma(\mu)} d\mu \frac{\partial}{\partial \mu} \left(\int_\mu^x \frac{\exp(i\lambda(x-\xi+\mu))}{\Delta(\lambda)} f(\xi) d\xi \right)
\end{aligned}$$

This proves the lemma.

Lemma 2.1 The convolution, defined by the formula

$$(g * f)(x) := i \int_0^x g(x-\xi) f(\xi) d\xi - \int_0^b \overline{\sigma(\mu)} d\mu \frac{\partial}{\partial \mu} \left(\int_\mu^x g(x-\xi+\mu) f(\xi) d\xi \right)$$

for $g, f \in W_2^1[0, b]$ is bilinear, commutative and associative.

Proof. Let us introduce an operation \circ as

$$(g \circ f)(x, t) := \int_t^x g(x-\xi+\mu) f(\xi) d\xi,$$

then we can rewrite the expression $g * f$ in the form

$$\begin{aligned}
(g * f) &= i(g \circ f)(x, 0) - \int_0^b \frac{\partial}{\partial \mu} (g \circ f)(x, \mu) \overline{\sigma(\mu)} d\mu = \\
&= i[(g \circ f)(x, 0) - \int_0^b (-i \frac{\partial}{\partial \mu} (g \circ f)(x, \mu)) \overline{\sigma_0(\mu)} d\mu] = iU_t(g \circ f)(x, t),
\end{aligned}$$

where

$$U(y) = y(0) - \int_0^b (-i \frac{\partial}{\partial \mu}) \overline{\sigma(\mu)} d\mu.$$

Hence, bilinearity is obvious. It is easy to see that the operation \circ is commutative. Indeed

$$\begin{aligned}
(g \circ f)(x, t) &= \int_t^x g(x+t-\xi) f(\xi) d\xi = \begin{pmatrix} \xi' = x+t-\xi \\ d\xi = -d\xi' \\ x \rightarrow t \\ t \rightarrow x \end{pmatrix} = \\
&= - \int_x^t g(\xi') f(x+t-\xi') d\xi' = \int_t^x f(x+t-\xi) g(\xi) d\xi = (f \circ g)(x, t)
\end{aligned}$$

i.e.

$$(g \circ f)(x, t) = (f \circ g)(x, t)$$

Then the operation $*$ commutative too,

$$(g * f)(x) = i(g \circ f)(x, 0) - \int_0^b \frac{\partial}{\partial \mu} (g \circ f)(x, \mu) \overline{\sigma(\mu)} d\mu =$$

$$= i(f \circ g)(x, 0) - \int_0^b \frac{\partial}{\partial \mu} (f \circ g)(x, \mu) \overline{\sigma(\mu)} d\mu = (f * g)(x)$$

Remark 2.1 The convolution is expressed in terms of boundary conditions and has the form

$$(f * g)(x) = iU_\mu \left\{ \int_\mu^x f(\xi) g(x + \mu - \xi) d\xi \right\}, \quad (4)$$

where

$$U(y) := y(0) - \int_0^b \left(-i \frac{dy}{dx}\right) \overline{\sigma(x)} dx.$$

Remark 2.2 From (3) it is easy to see that the resolvent in terms of convolution is represented in the form

$$(L - \lambda I)^{-1} f = \frac{\exp(i\lambda x)}{\Delta(\lambda)} * f(x). \quad (5)$$

Lemma 2.2 For any f from the domain D_L and for arbitrary g from $W_2^1[0, b]$, the equality

$$\frac{d}{dx} (f * g) = \frac{df}{dx} * g$$

holds true.

Proof. Since the equality $U(f) = 0$ holds for $f \in D_L$, we get

$$\begin{aligned} \frac{d}{dx} (f * g) &= if(0)g(x) + i \int_0^x \frac{df(x - \xi)}{dx} g(\xi) d\xi - \\ &- \int_0^b \overline{\sigma(\mu)} d\mu \frac{\partial}{\partial \mu} \left(f(\mu)g(x) + \int_\mu^x \frac{\partial}{\partial x} f(x - \xi + \mu) g(\xi) d\xi \right) = \frac{df}{dx} * g + if(0)g(x) - \\ &- \int_0^b \overline{\sigma_0(x)} \frac{\partial}{\partial \mu} f(\mu) g(x) d\mu = \frac{df}{dx} * g + ig(x)U(f) = \frac{df}{dx} * g. \end{aligned}$$

Lemma 2.3 The convolution defined in Lemma 2.1 is without annihilators, i.e. if for arbitrary

$$g \in W_2^1[0, b]$$

and for some $f \in L_2(0, b)$ the equality

$$(g * f)(x) = 0$$

holds, then f is identically equal to zero.

Proof. Let $(g * f)(x) = 0$, $f \in L_2(0, b)$, then we take

$$g =: \frac{\exp(i\lambda x)}{\Delta(\lambda)}|_{\lambda=0} = 1.$$

By the definition of our convolution, the expression $1 * f = 0$ denotes the equality $L^{-1}f = 0$. Denoting $L^{-1}f$ by y , we obtain respectively $Ly = f$, but $y = 0$, hence, as the L a linear operator, it follows that $f = 0$.

Lemma 2.4 If the function f is from the domain D_L of the operator L , then for each $g \in L_2(0, b)$ the convolution $f * g$ will also belong to the domain D_L .

Proof. Let

$$f =: \frac{\exp(i\lambda_0 x)}{\Delta(\lambda_0)}|_{\lambda_0=0} = 1.$$

Let us denote by y the function $y = 1 * g$, by construction of the convolution this expression means that $y = L^{-1}g$ i.e. $Ly = g$, which implies $y \in D_L$. Now we fix any λ such that $\Delta(\lambda) \neq 0$. Let $f =: \frac{\exp(i\lambda x)}{\Delta(\lambda)}$.

If by y we denote $y = 1 * g$, then this expression by the convolution construction mean that $y = L^{-1}g$, i.e. $Ly = g$, and implies $y \in D_L$. Now we fix any λ , such that $\Delta(\lambda) \neq 0$. Let $f =: \frac{\exp(i\lambda x)}{\Delta(\lambda)}$. Let y denote the convolution $y = \frac{\exp(i\lambda x)}{\Delta(\lambda)} * g$, then from the definition of the convolution, we get $y = (L - \lambda I)^{-1}g$, i.e. $Ly = \lambda y + g$ for any λ . Since zeros of the function $\Delta(\lambda)$ are countable set, then there is sequence $\{\lambda_n\}$ such that $\Delta(\lambda_n) \neq 0$ and the system $\exp(i\lambda_n x)$ is basis in the space $W_2^1[0, b]$. For any natural number n it is λ_n such that the convolution $\exp(i\lambda_n x) * g$ will belong to the domain D_L . From $f(x) = \sum_{n=0}^{\infty} c_n \exp(i\lambda_n x)$ and from the bilinearity of the convolution, the convolution $f * g$ belongs to the domain D_L .

Lemma 2.5 For arbitrary λ and β such that $\lambda \neq \beta$ we have the equality

$$\exp(i\lambda x) * \exp(i\beta x) = \frac{\exp(i\beta x)\Delta(\lambda) - \exp(i\lambda x)\Delta(\beta)}{\beta - \lambda}.$$

Proof. We write by definition

$$\begin{aligned} \exp(i\lambda x) * \exp(i\beta x) &= \\ &= i \int_0^x \exp(i\lambda(x - \xi)) \exp(i\beta \xi) d\xi - \\ &- \int_0^b \overline{\sigma(\mu)} d\mu \frac{\partial}{\partial \mu} \left(\int_{\mu}^x \exp(i\lambda(x - \xi + \mu)) \exp(i\beta \xi) d\xi \right) = \\ &= \frac{i \exp(i\lambda x) (\exp(i(\beta - \lambda)x) - 1)}{i(\beta - \lambda)} - \\ &- \int_0^b \overline{\sigma(\mu)} d\mu \frac{\partial}{\partial \mu} \left(\exp(i\lambda(x + \mu)) \frac{\exp(i(\beta - \lambda)x) - \exp(i(\beta - \lambda)\mu)}{i(\beta - \lambda)} \right) = \\ &= \frac{\exp(i\beta x) - \exp(i\lambda x)}{\beta - \lambda} - \\ &- \int_0^b \overline{\sigma(\mu)} d\mu \frac{\partial}{\partial \mu} \left[\frac{\exp(i\lambda \mu) \exp(i\beta x) - \exp(i\lambda x) \exp(i\beta \mu)}{i(\beta - \lambda)} \right] = \\ &= \frac{\exp(i\beta x) - \exp(i\lambda x)}{\beta - \lambda} - \\ &- \int_0^b \frac{i\lambda \exp(i\lambda \mu) \exp(i\beta x) - i\beta \exp(i\lambda x) \exp(i\beta \mu)}{i(\beta - \lambda)} \overline{\sigma(\mu)} d\mu = \\ &\frac{1}{\beta - \lambda} [\exp(i\beta x) - \exp(i\lambda x) - \lambda \exp(i\beta x) \int_0^b \exp(i\lambda \mu) \overline{\sigma(\mu)} d\mu + \end{aligned}$$

$$+\beta \exp(i\lambda x) \int_0^b \exp(i\beta\mu) \overline{\sigma(\mu)} d\mu] = \frac{\exp(i\beta x)\Delta(\lambda) - \exp(i\lambda x)\Delta(\beta)}{\beta - \lambda}.$$

3. Fourier transform and convolution

For any function f from the space $L_2(0, b)$, let us associate the expansion

$$f \sim \sum_{\lambda_n \in \sigma(L)} P_n f,$$

where the orthogonal projection is given by

$$P_n f = -\frac{1}{2\pi i} \oint_{|\lambda - \lambda_n| = \delta} (L - \lambda I)^{-1} f d\lambda.$$

By using (5) we obtain

$$\begin{aligned} P_n f &= \operatorname{res}_{\lambda_n} \frac{\exp(i\lambda x)}{\Delta(\lambda)} * f = \left[\frac{1}{(m_n - 1)!} \lim_{\lambda \rightarrow \lambda_n} \frac{\partial^{m_n-1}}{\partial \lambda^{m_n-1}} \left(\frac{(\lambda - \lambda_n)^{m_n}}{\Delta(\lambda)} \exp(i\lambda x) \right) \right] * f(x) = \\ &= \left[\sum_{j=0}^{m_n-1} \frac{d_{j,n}}{j!} \left(\frac{(ix)^{m_n-1-j}}{(m_n - 1 - j)!} \exp(i\lambda_n x) \right) \right] * f(x). \end{aligned}$$

Let us define

$$u_{m_n-1,n} := \sum_{j=0}^{m_n-1} \frac{d_{j,n}}{j!} \left(\frac{(ix)^{m_n-1-j}}{(m_n - 1 - j)!} \exp(i\lambda_n x) \right), \quad (6)$$

where $\{d_{j,n}\}$ are the expansion coefficients in the Taylor series of the function $\frac{(\lambda - \lambda_n)^{m_n}}{\Delta(\lambda)}$ in front of powers of $(\lambda - \lambda_n)$. Then

$$P_n f = u_{m_n-1,n} * f. \quad (7)$$

Let us introduce a system of root functions of the operator L corresponding to the eigenvalue λ_n by the expression

$$u_{s,n} := L u_{s+1,n} - \lambda_n u_{s+1,n} = \sum_{j=0}^s \frac{d_{j,n}}{j!} \frac{(ix)^{s-j}}{(s-j)!} \exp(i\lambda_n x), \quad s = 0, \dots, m_n - 2. \quad (8)$$

Lemma 3.1 The system of root functions $\{u_{k,n}, k = 0, \dots, m_n - 1\}$ is linearly independent.

Proof. Consider the linear combination

$$\alpha_0 u_{0,n} + \dots + \alpha_{m_n-2} u_{m_n-2,n} + \alpha_{m_n-1} u_{m_n-1,n} = 0. \quad (9)$$

It is easy to see that

$$(L - \lambda_n I)^{m_n-1} u_{s,n} = 0, \quad s = 0, \dots, m_n - 2. \quad (10)$$

By applying to (9) the operator $(L - \lambda_n I)^{m_n-1}$ on both sides, we get

$$\alpha_{m_n-1}(L - \lambda_n I)^{m_n-1} u_{m_n-1,n} = 0. \quad (11)$$

From (8) we see that (11) is equivalent to

$$\alpha_{m_n-1} u_{0,n} = 0.$$

Since $d_{0,n} \neq 0$, we obtain

$$\alpha_{m_n-1} = 0.$$

Now we consider the equality (9) with $\alpha_{m_n-1} = 0$, i.e.

$$\alpha_0 u_{0,n} + \dots + \alpha_{m_n-2} u_{m_n-2,n} = 0.$$

Applying to both sides the operator $(L - \lambda_n I)^{m_n-2}$ and using (10) as in the previous step, we get

$$\alpha_{m_n-2} u_{0,n} = 0,$$

whence $\alpha_{m_n-2} = 0$.

Repeating these steps, we find that equality (9) is true only for $\alpha_k = 0, k = 0, \dots, m_n - 1$. This proves the linear independence of the system $\{u_{k,n}, k = 0, \dots, m_n - 1\}$.

Since $d_{0,n} \neq 0$, the system $u_{0,n}, u_{1,n}, \dots, u_{m_n-1,n}$ is linearly independent and, therefore, there is a basis of the root subspace

$$H_{\lambda_n} := \text{Ker}(L - \lambda_n I)^{m_n}.$$

We expand the function $P_n f$ on $\text{Ker}(L - \lambda_n I)^{m_n}$ by this basis

$$P_n f = C_{m_n-1,n}(f) u_{0,n} + C_{m_n-2,n}(f) u_{1,n} + \dots + C_{0,n}(f) u_{m_n-1,n}.$$

Thus, for each element f of the space $L_2(0, b)$ we associate the element of the sequence space

$$\{(C_{0,n}(f), \dots, C_{m_n-1,n}(f)), \lambda_n \in \sigma(L)\} \in \prod_{\lambda_n \in \sigma(L)} \mathbb{C}^{m_n},$$

i.e. we introduce the Fourier transform

$$\widehat{f} = \{(C_{0,n}(f), \dots, C_{m_n-1,n}(f)), \lambda_n \in \sigma(L)\}.$$

In the space of sequences, we introduce the inner Cauchy convolution. Let ξ and η be any elements of the $X := \prod_{\lambda_n \in \sigma(L)} \mathbb{C}^{m_n}$, then we will call their convolution the sequence

$$\mu := \{(\xi_{0,n} \eta_{0,n}; \xi_{0,n} \eta_{1,n} + \xi_{1,n} \eta_{0,n}; \dots, \sum_{k=0}^{m_n-1} \xi_{k,n} \eta_{m_n-1-k,n}), \lambda_n \in \sigma(L)\},$$

which we denote by $\xi *_X \eta$. Introduced convolutions $*_X$ and $*$ are associated between themselves by the Fourier transform.

Theorem 3.1 For arbitrary functions f and g from the space $W_2^1[0, b]$ the equality

$$\widehat{f * g} = \widehat{f} *_X \widehat{g} \quad (12)$$

holds.

Proof. Let $f = u_{m_n-1,n}$. Then the equality $\widehat{u_{m_n-1,n} * g} = \widehat{u_{m_n-1,n}} *_X \widehat{g}$ follows from the following chain of equalities

$$\begin{aligned} (u_{m_n-1,n} * g)^\wedge &= (P_n g)^\wedge = (C_{0,n}(g), C_{1,n}(g), \dots, C_{m_n-1,n}(g)) = \\ &= (1, 0, \dots, 0) *_X (C_{0,n}(g), C_{1,n}(g), \dots, C_{m_n-1,n}(g)). \end{aligned}$$

Let now $f = u_{s,n}$, then

$$\begin{aligned} (u_{s,n} * g)^\wedge &= ((L - \lambda_n I)^{m_n-1-s} u_{m_n-1,n} * g)^\wedge = \\ &= ((L - \lambda_n I)^{m_n-1-s} (u_{m_n-1,n} * g))^\wedge = ((L - \lambda_n I)^{m_n-1-s} P_n g)^\wedge = (P_n (L - \lambda_n I)^{m_n-1-s} g)^\wedge = \\ &= (C_{0,n}(L - \lambda_n I)^{m_n-1-s} g), (C_{1,n}(L - \lambda_n I)^{m_n-1-s} g), \dots, (C_{m_n-1,n}(L - \lambda_n I)^{m_n-1-s} g). \end{aligned}$$

On the other hand, the relations

$$\begin{aligned} C_{m_n-1,n}((L - \lambda_n I)g) &= C_{m_n-2,n}(g), \\ &\dots, \\ C_{1,n}((L - \lambda_n I)g) &= C_{0,n}(g) \end{aligned}$$

and

$$C_{0,n}((L - \lambda_n I)g) = 0$$

are valid, i.e. the action of the operator $(L - \lambda_n I)$ is equivalent to a shift in the sequence space. Then action of the operator $(L - \lambda_n I)^{m_n-1-s}$ corresponds to a shift to the right on $m_n - 1 - s$ position. This implies the equality

$$\widehat{u_{s,n} * g} = \widehat{u_{s,n}} *_X \widehat{g}, \quad s = 0, \dots, m_n - 1.$$

From the fact that the theorem holds for all elements of the basis $\{u_{s,n}\}$ follows the required equality (12) follow in the whole space.

4. Coefficient functionals and a boundary condition

Elements of the basis of the each root subspace have good properties.

Lemma 4.1 Let λ_n be zeros of the entire function $\Delta(\lambda)$ with corresponding multiplicities m_n . Then the elements (6) and (8) of the root subspace H_{λ_n} have the following properties:

$$u_{p,n} * u_{q,n} = \begin{cases} 0, & \text{for } p + q < m_n - 1, \\ u_{p+q-m_n+1}, & \text{for } p + q \geq m_n - 1, \end{cases} \quad (13)$$

$0 \leq p, q \leq m_n - 1$.

Proof. At first, we note that the element (6) is idempotent with respect to the convolution, i.e.

$$u_{m_n-1,n} * u_{m_n-1,n} = u_{m_n-1,n}.$$

By acting on the element $u_{m_n-1,n}$ of the basis of root subspace of the operator P_n , we get

$$P_n(u_{m_n-1,n}) = u_{m_n-1,n}.$$

On the other hand, by replacing the function f by $u_{m_n-1,n}$ in the formula (7), we have

$$P_n(u_{m_n-1,n}) = u_{m_n-1,n} * u_{m_n-1,n},$$

which proves the property that the element $u_{m_n-1,n}$ is idempotent. Further, from (8) it is easy to see that each basis element can be presented by an idempotent element $u_{m_n-1,n}$ in the form

$$u_{q,n} = (L - \lambda_n I)^{m_n-1-q} u_{m_n-1,n}, q = 0, \dots, m_n - 2. \quad (14)$$

Then consider the convolution of two elements of the basis

$$u_{q,n} * u_{p,n} = [(L - \lambda_n I)^{m_n-1-q} u_{m_n-1,n}] * [(L - \lambda_n I)^{m_n-1-p} u_{m_n-1,n}] =$$

|by using bilinear property of the convolution $*$ |

$$= (L - \lambda_n I)^{2m_n-2-q-p} (u_{m_n-1,n} * u_{m_n-1,n}) =$$

|by using idempotence property of the element $u_{m_n-1,n}$ |

$$= (L - \lambda_n I)^{m_n-1-(q+p-m_n+1)} u_{m_n-1,n}.$$

It is clear that if the inequality $q + p - m_n + 1 \geq 0$ holds, then from (14) it follows that

$$u_{p,n} * u_{q,n} = u_{p+q-m_n+1,n}$$

and otherwise

$$u_{p,n} * u_{q,n} = 0.$$

In the following lemma we give the coefficient functionals in the boundary condition.

Lemma 4.2 The coefficient functionals in the projector expansion $P_n f = \sum_{k=0}^{m_n-1} C_{m_n-1-k,n}(f) u_{k,n}$ in the root subspace $\text{Ker}(L - \lambda_n I)^{m_n}$ are of the form

$$C_{k,n}(f) = -iU_\mu \left\{ \int_0^\mu f(\xi) \frac{(i(\mu - \xi))^k}{k!} \exp(i\lambda_n(\mu - \xi)) d\xi \right\}, 0 \leq k \leq m_n - 1.$$

Proof. By the definition we have $P_n f = f * u_{m_n-1,n}$, but on other hand $P_n f = C_{m_n-1,n}(f) u_{0,n} + C_{m_n-2,n}(f) u_{1,n} + \dots + C_{0,n}(f) u_{m_n-1,n}$. Then we get the equality

$$f * u_{m_n-1,n} = \sum_{k=0}^{m_n-1} C_{m_n-1-k,n}(f) u_{k,n}, \quad (15)$$

As for $u_{0,n}$, we have

$$f * u_{m_n-1,n} * u_{0,n} = \sum_{k=0}^{m_n-1} C_{m_n-1-k,n}(f) u_{k,n} * u_{0,n},$$

then by using Lemma 3.1, we get

$$f * u_{0,n} = C_{0,n}(f)u_{0,n}. \quad (16)$$

Let rewrite the left side of (16) by using formula (4)

$$\begin{aligned} f * u_{0,n} &= -iU_\mu \left\{ \int_\mu^x f(\xi)u_{0,n}(x + \mu - \xi)d\xi \right\} = \\ &= -iU_\mu \left\{ \int_0^x f(\xi)u_{0,n}(x + \mu - \xi)d\xi \right\} - iU_\mu \left\{ \int_0^\mu f(\xi)u_{0,n}(x + \mu - \xi)d\xi \right\}. \end{aligned}$$

Since $iU_\mu \left\{ \int_0^x f(\xi)d_{0,n} \exp(i\lambda_n(x + \mu - \xi))d\xi \right\} =$

$$\begin{aligned} &= d_{0,n} \exp(i\lambda_n x) i \int_0^x f(\xi) \exp(-i\lambda_n \xi) d\xi [1 - \lambda_n \int_0^b \exp(i\lambda_n \mu) \overline{\sigma(\mu)} d\mu] = \\ &= u_{0,n} i \Delta(\lambda_n) \int_0^x f(\xi) \exp(-i\lambda_n \xi) d\xi, \end{aligned}$$

the first term is equal to zero. And

$$\begin{aligned} f * u_{0,n} &= -iU_\mu \left\{ \int_0^\mu f(\xi)u_{0,n}(x + \mu - \xi)d\xi \right\} = -iU_\mu \left\{ \int_0^\mu f(\xi)d_{0,n} \exp(i\lambda_n(x + \mu - \xi))d\xi \right\} = \\ &= -d_{n,0} \exp(-i\lambda_n \xi) iU_\mu \left\{ \int_0^\mu f(\xi) \exp(i\lambda_n(\mu - \xi))d\xi \right\} = -iU_\mu \left\{ \int_0^\mu f(\xi) \exp(i\lambda_n(\mu - \xi))d\xi \right\} u_{0,n}. \end{aligned}$$

By comparing the right and the left hand side of (16), we get

$$C_{0,n}(f) = -iU_\mu \left\{ \int_0^\mu f(\xi) \exp(i\lambda_n(\mu - \xi))d\xi \right\}.$$

Now suppose that

$$C_{k,n}(f) = -iU_\mu \left\{ \int_0^\mu f(\xi) \frac{(i(\mu - \xi))^k}{k!} \exp(i\lambda_n(\mu - \xi))d\xi \right\}, k = 0 : s-1, 1 \leq s \leq m_n - 1.$$

Let us prove that

$$C_{s,n}(f) = -iU_\mu \left\{ \int_0^\mu f(\xi) \frac{(i(\mu - \xi))^s}{s!} \exp(i\lambda_n(\mu - \xi))d\xi \right\}, 1 \leq s \leq m_n - 1.$$

By convolving the both sides of (15) with $u_{s,n}$, we have

$$f * u_{m_n-1,n} * u_{s,n} = \sum_{k=0}^{m_n-1} C_{m_n-1-k,n}(f)u_{k,n} * u_{s,n}. \quad (17)$$

The equality (17) is changed to

$$f * u_{s,n} = \sum_{k=m_n-1-s}^{m_n-1} C_{m_n-1-k,n}(f)u_{k+s-m_n+1,n}.$$

Let us related the index k by $l = m_n - 1 - k$, then we get

$$f * u_{s,n} = \sum_{l=0}^s C_{l,n}(f) u_{s-1,n}. \quad (18)$$

As in the previous case by using formula (4), we write

$$\begin{aligned} f * u_{s,n} &= iU_\mu \left\{ \int_\mu^x f(\xi) u_{s,n}(x + \mu - \xi) d\xi \right\} = \\ &= iU_\mu \left\{ \int_0^x f(\xi) u_{s,n}(x + \mu - \xi) d\xi \right\} - iU_\mu \left\{ \int_0^\mu f(\xi) u_{s,n}(x + \mu - \xi) d\xi \right\}. \end{aligned}$$

Since

$$\begin{aligned} &iU_\mu \left\{ \int_0^x f(\xi) \sum_{j=0}^s \frac{d_{j,n}}{j!} \frac{(i(x + \mu - \xi))^{s-j}}{(s-j)!} \exp(i\lambda_n(x + \mu - \xi)) d\xi \right\} = \\ &= \sum_{j=0}^s \frac{d_{j,n}}{j!} \exp(i\lambda_n x) \sum_{p=0}^{s-j} \frac{(ix)^{s-j-p}}{(s-j-p)!} iU_\mu \left\{ \int_0^x f(\xi) \frac{(i(\mu - \xi))^p}{p!} \exp(i\lambda_n(\mu - \xi)) d\xi \right\} = \\ &= \sum_{j=0}^s \frac{\partial^j}{\partial \lambda^j} G(f; x, \lambda) |_{\lambda=\lambda_n} u_{s-j,n} \end{aligned}$$

the first term is equal to zero, here

$$G(f; x, \lambda) := i\Delta(\lambda) \int_0^x f(\xi) \exp(-i\lambda\xi) d\xi.$$

It is obvious that if λ_n is zero of the function $\Delta(\lambda)$, then $\frac{\partial^j}{\partial \lambda^j} G(f; x, \lambda) |_{\lambda=\lambda_n} = 0, \forall j \in Z_+$. In the left hand side of (18), we then have

$$\begin{aligned} f * u_{s,n} &= -iU_\mu \left\{ \int_0^\mu f(\xi) u_{s,n}(x + \mu - \xi) d\xi \right\} = \\ &= -iU_\mu \left\{ \int_0^\mu f(\xi) \sum_{j=0}^s \frac{d_{j,n}}{j!} \frac{(i(x + \mu - \xi))^{s-j}}{(s-j)!} \exp(i\lambda_n(x + \mu - \xi)) d\xi \right\} = \\ &= - \sum_{j=0}^s \frac{d_{j,n}}{j!} \exp(i\lambda_n x) \sum_{p=0}^{s-j} \frac{(ix)^{s-j-p}}{(s-j-p)!} iU_\mu \left\{ \int_0^\mu f(\xi) \frac{(i(\mu - \xi))^p}{p!} \exp(i\lambda_n(\mu - \xi)) d\xi \right\} = \\ &= \sum_{j=0}^s C_{j,n}(f) u_{s-j,n} - iU_\mu \left\{ \int_0^\mu f(\xi) \frac{(i(\mu - \xi))^s}{s!} \exp(i\lambda_n(\mu - \xi)) d\xi \right\} u_{0,n}. \end{aligned}$$

By comparing the right and the left hand sides of the equality (18), we get

$$C_{s,n}(f) = -iU_\mu \left\{ \int_0^\mu f(\xi) \frac{(i(\mu - \xi))^s}{s!} \exp(i\lambda_n(\mu - \xi)) d\xi \right\}.$$

Hence we have explicitly constructed a biorthogonal system to the system $\{u_{k,n}, \lambda_n \in \sigma(L)\}$.

Theorem 4.2 The chosen basis system $\{u_{k,n}, k = 0, \dots, m_n - 1, \lambda_n \in \sigma(L)\}$ is minimal in $L_2(0, b)$, i.e. there exists a biorthogonal system of the form

$$h_{k,n}(\xi) = \int_{\xi}^b \overline{\sigma(\mu)} \frac{\partial}{\partial \mu} \left(\frac{(i(\mu - \xi))^k}{k!} \exp(i\lambda_n(\mu - \xi)) \right) d\mu, k = 0, \dots, m_n - 1.$$

5. On the Sedletski formula for the remainder term

It is well-known that the partial sum of the Fourier series is written in the form

$$S_R(f; x) = -\frac{1}{2\pi i} \oint_{|\lambda|=R} (L - \lambda I)^{-1} f d\lambda = \sum_{|\lambda_n| < R} P_{\lambda_n} f.$$

By using (5), we rewrite the partial sum as

$$S_R(f; x) = -\frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{\exp(i\lambda x)}{\Delta(\lambda)} * f(x) d\lambda.$$

In this formula instead of f we put the function $\exp(i\mu x)$. Then we have

$$S_R(\exp(i\mu x); x) = -\frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{\exp(i\lambda x)}{\Delta(\lambda)} * \exp(i\mu x) d\lambda =$$

|by applying lemma 2.5| =

$$\begin{aligned} &= \frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{\exp(i\lambda x) \Delta(\mu) - \exp(i\mu x) \Delta(\lambda)}{\Delta(\lambda)(\mu - \lambda)} d\lambda = \\ &= \frac{\Delta(\mu)}{2\pi i} \oint_{|\lambda|=R} \frac{\exp(i\lambda x)}{\Delta(\lambda)} \frac{d\lambda}{\mu - \lambda} + \frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{\exp(i\mu x)}{\lambda - \mu} d\lambda = \end{aligned}$$

|by using Cauchy formula to the second term| =

$$= \exp(i\mu x) + \frac{\Delta(\mu)}{2\pi i} \oint_{|\lambda|=R} \frac{\exp(i\lambda x)}{\Delta(\lambda)} \frac{d\lambda}{\mu - \lambda}.$$

Hence it follows that the formula for the remainder term for $f(x) = \exp(i\mu x)$ is

$$Q_R(\exp(i\mu x); x) := S_R(\exp(i\mu x); x) - \exp(i\mu x) = \frac{\Delta(\mu)}{2\pi i} \oint_{|\lambda|=R} \frac{\exp(i\lambda x)}{\Delta(\lambda)} \frac{d\lambda}{\mu - \lambda}, \quad (19)$$

which is valid for any μ . Since an arbitrary element f of the space $L_2(0, b)$ is represented in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(-\mu) \exp(i\mu x) d\mu,$$

where $\hat{f}(\mu)$ is the Fourier transform of the function $f(x)$, from (19), we obtain the integral form of the remainder term for an arbitrary function $f(x)$ in the space $L_2(0, b)$,

$$Q_R(f; x) := S_R(f(x); x) - f(x) = \frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{\exp(i\lambda x)}{\Delta(\lambda)} d\lambda \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{f}(-\mu) \Delta(\mu)}{\mu - \lambda} d\mu \right).$$

A similar formula was proved in another way by A.M. Sedletskiy in the work [6]. Let us write the set of conditions

$$\begin{cases} \max \supp \sigma(x) = b, \\ \min \supp (\sigma(x) + i) = 0, \end{cases} \quad (A)$$

$$\sup |Im \lambda_n| = M < \infty, \quad (B)$$

$$\inf_{n \neq k} |\lambda_n - \lambda_k| > 0, \quad (C)$$

$$\sup m_n = m < \infty. \quad (D)$$

Denote $\omega(\lambda) = |\Delta(\lambda)|^2$. Let us write the Muckenhoupt condition

$$\sup_I \left(\frac{1}{|I|} \int_I \omega(\lambda) d\lambda \right) \left(\frac{1}{|I|} \int_I \omega^{-1}(\lambda) d\lambda \right) < \infty, \quad (E)$$

where I an arbitrary interval of the real axis. If the operator

$$S^+ : \sum_{\lambda_n \in \Lambda} P_n(x) \exp(i\lambda_n x) \mapsto \sum_{Re \lambda_n > 0} P_n(x) \exp(i\lambda_n x)$$

is bounded in the space $L_2(0, b)$, then we will say that the basis $\{u_{k,n}, k = 0, \dots, m_n - 1, \lambda_n \in \sigma(L)\}$ of the space $L_2(0, b)$ is Riesz basis.

The following theorems are true.

Theorem 5.1 Let the conditions (A), (B), (C), (D) be valid and assume that the function $\omega(\lambda) = |\Delta(\lambda)|^2$ satisfies the Muckenhoupt condition (E). Then the system $\{u_{k,n}, k = 0, \dots, m_n - 1, \lambda_n \in \sigma(L)\}$ is a Riesz basis in the space $L_2(0, b)$.

Theorem 5.2 Assume the conditions of Theorem 5.1, except, perhaps, the condition (D). Then for every $f \in L_2(0, b)$, we have

$$\|x^{\frac{1}{2}}(b-x)^{\frac{1}{2}} Q_r(f, x)\|_{C[0,b]} \rightarrow 0$$

continuously for $r \rightarrow \infty$.

Theorem 5.3 Suppose the conditions of Theorem 5.1. Then for every $f \in L_2(0, b)$ the coefficient sequence satisfies

$$\{\{c_{k,n}\}_{k=0}^{\infty}\}_{n=0}^{\infty} \in l_2,$$

and $\|\{c_{k,n}\}\|_{l_2} \leq C(L)\|f\|_{L_2}$.

REFERENCE

- [1] M.A. Naimark, Linear differential operators, GITTL, Moscow, 1954; English transl., Part II, Ungar, New York, 1968.
- [2] M. Ruzhansky, V. Turunen, Pseudo-Differential Operators and Symmetries, Birkhauser, 2010.
- [3] B.E. Kanguzhin and M.A. Sadybekov, Differential operators on a segment. Distributions of the eigenvalues, Gylym, Almaty, 1996. [in russian].
- [4] E.C. Titchmarsh, The zeros of certain integral functions // Proc. London Math. Soc. 1926, V. 25, No 4, pp. 283–302.
- [5] M.L. Cartwright, The zeros of certain integral functions // The Quarterly Journal of Math., Oxford series, 1930, V. 1, No 1, pp. 38–59.
- [6] A.M. Sedletskiy, On biorthogonal expansions of functions to exponent series on a real segment // Usp. Mat. Nauk, 1982, V. 37, No 5(227), pp. 51–95 [in russian].